Bayesian Analysis of RR Lyrae Distances and Kinematics

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Abstract

- We have developed a hierarchical Bayes model to analyze the distances, luminosities, and kinematics of RR Lyrae stars. Our model relates these parameters to the observed proper motions, radial velocities, apparent luminosities and metallicities of the stellar sample. We use a Metropolis-within-Gibbs sampler to draw an MCMC sample from the full posterior distribution of the parameters (including latent variables), and draw inferences on the quantities of interest in the usual way. We are testing our model with a small database from the literature and will eventually apply it to a new large database from the European HIPPARCOS satellite.

Outline

- A little astronomy—what's the goal?
- Mathematical model and likelihood function
- Priors
- Sampling strategy
- Results
- Future research

Description of the Problem

- RR Lyrae stars are a class of pulsating variable stars. They are readily recognizable from their periods 0.75±0.25 days) and characteristic light curves.
- They are old (evolved) stars with distinctive kinematics (statistical description of their motions in the galaxy)

Description of the Problem

- They are fairly bright (40 times as bright as the Sun) and so can be seen to fair distances in the galaxy.
- They have the useful property that their intrinsic visual-band luminosities are nearly constant.
  - This is known from studies of RR Lyrae stars in clusters, where all the stars are at the same distance.
- These characteristics make these stars useful as “standard candles” for estimating distances in the universe: We observe their apparent luminosities, know their intrinsic luminosities, and use the fact that the apparent luminosity falls off as the square of the distance.

Goals of Our Investigation

- Determine the absolute magnitude (log luminosity) of these stars
- Investigate any variation of absolute magnitude with "metallicity" (i.e., content of elements heavier than helium)
- Investigate the "cosmic scatter" of the magnitude (i.e., the variation about the mean unexplained by other variables)
- Investigate the kinematics of the stars as a group (i.e., it is believed that the velocities of these stars are distributed roughly with a multivariate normal distribution; what are the parameters of that distribution?)

How We Do It

- Our raw data are the proper motions $\boldsymbol{\mu}$ (vector of angular motion/time unit of the motions of the stars across the sky), the radial velocities ρ (kilometers/second of the motions towards or away from the Sun, obtained by Doppler shift), and the apparent magnitudes m of the stars, assumed measured without error.
- The proper motions are related to the cross-track velocities (in km/sec) by multiplying the former by the distance s to the star:

$$s\boldsymbol{\mu} \propto \mathbf{V}^{\perp}$$

By assuming that the proper motions and radial velocities are characterized by the same kinematical parameters, we can (statistically) infer s and then the magnitude M of the star through a known relationship:

$$s = 10^{0.2(m-M+5)}$$

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The Likelihood

- If $\mu^o=(\mu_\alpha{}^o, \mu_\delta{}^o)$ are the two components of the *observed* proper motion vector (in the plane of the sky), and ρ^o is the radial velocity, we assume

$$\mu_\alpha^o \sim N(\mu_\alpha, \sigma^2_{\mu_\delta})$$

$$\mu_\delta^o \sim N(\mu_\delta, \sigma^2_{\mu_\delta})$$

$$\rho^o \sim N(V^{\parallel}, \sigma^2_\rho)$$

- The variances are estimated at the telescope and are assumed *known perfectly*. The variables without superscripts are the "true" values.

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Quantities Being Estimated

- The posterior distribution is a function of at least the following quantities:
  - The true velocities $\mathbf{V}_i$ of the stars
  - The velocity $\mathbf{V_0}$ of the Sun relative to the mean of the stars
  - The three-dimensional covariance matrix $\mathbf{W}$ that describes the kinematics of the stellar velocities
  - The distance s_i to each star, and each star's associated absolute magnitude M_i
  - The mean absolute magnitude M of the RR Lyrae stars as a group
- In addition, we will introduce several additional variables for convenience.

Priors Defining the Hierarchical Model

- Temporarily suppressing the subscript i, we enforce the relationship $s\boldsymbol{\mu} \propto \mathbf{V}^{\perp}$ with the delta-function "prior"

$$\pi(\boldsymbol{\mu} \mid \mathbf{V}, s) = \delta(\mu_\alpha - V_\alpha^{\perp}/s)\delta(\mu_\delta - V_\delta^{\perp}/s)$$

where simple linear algebra gives the relationships

$\hat{\mathbf{s}}$ is the unit vector towards the star,

$$\mathbf{V} = (V_\alpha^{\perp}, V_\delta^{\perp}, V^{\parallel}) = \mathbf{V}^{\perp} + \mathbf{V}^{\parallel} = \mathbf{V}^{\perp} + \hat{\mathbf{s}}V^{\parallel}$$

$\mathbf{V}^{\parallel} = \hat{\mathbf{s}}\hat{\mathbf{s}}'\mathbf{V}$ (projects $\mathbf{V}$ onto $\hat{\mathbf{s}}$)

$\mathbf{V}^{\perp} = \mathbf{V} - \mathbf{V}^{\parallel} = (\mathbf{I} - \hat{\mathbf{s}}\hat{\mathbf{s}}')\mathbf{V}$ (projects $\mathbf{V}$ onto plane of sky)

$V^{\parallel} = \hat{\mathbf{s}}'\mathbf{V}$

$\hat{\mathbf{s}}'\mathbf{V}^{\perp} = \mathbf{0}$

Priors Defining the Hierarchical Model

- The priors on the $\mathbf{V}_i$ are obtained by assuming that the velocities of the individual stars are drawn from a (three-dimensional) multivariate normal distribution:

$$\mathbf{V}_i \mid \mathbf{W}, \mathbf{V_0} \sim N(\mathbf{V_0}, \mathbf{W})$$

- We choose a flat prior on $\mathbf{V_0}$, and to avoid an improper posterior distribution, an "independence Jeffreys prior" on $\mathbf{W}$, which for a three-dimensional distribution implies

$$\pi(\mathbf{W}) \propto \mid \mathbf{W} \mid^{-1}$$

Priors Defining the Hierarchical Model

- We formally enforce the astronomical relationship between distance and magnitude with a delta-function "prior":

$$\pi(s_i \mid M_i) = \delta(s_i - 10^{0.2(m_i - M_i + 5)})$$

- We introduce new variables $U_i = M_i - M$, because preliminary calculations indicated that sampling would be more efficient on the variables U_i (the M_i are highly correlated with themselves and with M, but the U_i are not). For convenience, we introduce a delta-function prior on M_i:

$$\pi(M_i \mid M, U_i) = \delta(M_i - M - U_i)$$

- We choose the U_i as our actual parameters, with M_i derived.

Priors Defining the Hierarchical Model

- We choose a flat prior on M (we have also tried a somewhat informative prior based on known data).
- Evidence from other sources (e.g., studies of RR Lyrae stars in clusters) indicates a "cosmic scatter" of about 0.15 magnitudes in M_i. Thus a prior on U_i of the form

$$U_i \sim N(0,(0.15)^2)$$

seems appropriate.

The Role of the Delta-function "Priors"

- We introduced the delta-function "priors" to enforce exact relationships between several of the variables, which will assist us in formulating an effective sampling scheme. One should think of them as just that: A bookkeeping device to enforce exact relations between the primary variables $\boldsymbol{\mu}$ and M_i and secondary ones related to them such as $\mathbf{V}_i$ and s_i which may be convenient to use in our sampling scheme.
- For example, we may find it convenient to sample on $\mathbf{V}_i$, conditional on s_i. But $\mathbf{V}_i$ should be thought of as a proxy for the primary variables $\boldsymbol{\mu}$, and in this step immediately converted to them (using the current, fixed value of s_i).

The Role of the Delta-function Priors

- Similarly, when sampling on the M_i, the appearance of the corresponding s's in the delta-function "prior"

$$\pi(\boldsymbol{\mu} \mid \mathbf{V}^{\perp}, s) = \delta(\mu_{\alpha} - V_{\alpha}^{\perp}/s)\delta(\mu_{\delta} - V_{\delta}^{\perp}/s)$$

  introduces a factor s^2 in the joint distribution (when we integrate out the V variables)

- Since we derive the full conditionals from the joint distribution, we must take this into account.
- The factor s^2 is just the Jacobian of the transformation

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Sampling Strategy

- We can use Gibbs sampling to sample on $\mathbf{W}$ and $\mathbf{V_0}$

$$\mathbf{V_0} \mid \{\mathbf{V}_i\}, \mathbf{W} \sim N(\overline{\mathbf{V}}, \mathbf{W}/N)$$

where N is the number of stars in our sample and

$$\overline{\mathbf{V}} = \frac{1}{N}\sum \mathbf{V_i}$$

- Also,

$$\mathbf{W} \mid \{\mathbf{V}_i\}, \mathbf{V_0} \sim \text{InverseWishart}(\mathbf{T}, df = N - 1)$$

where

$$\mathbf{T} = \sum (\mathbf{V}_i - \mathbf{V_0})(\mathbf{V}_i - \mathbf{V_0})'$$

Sampling Strategy

- $\mathbf{V}_i$ can be sampled in a Gibbs step; it involves data on the individual stars as well as the general velocity distribution, and the covariance matrix $\mathbf{S}_i$ of the velocities of the individual stars depends on s_i because of the relation $s\boldsymbol{\mu} = \mathbf{V}^{\perp}$
- The resulting sampling scheme is

$$\mathbf{V}_i \sim N(\mathbf{u}_i, \mathbf{Z}_i)$$

where

$$\mathbf{Z}_i = (\mathbf{S}_i^{-1} + \mathbf{W}_i^{-1})^{-1}$$

$$\mathbf{u}_i = \mathbf{Z}_i(\mathbf{S}_i^{-1}\mathbf{V}_i^{\mathbf{o}} + \mathbf{W}^{-1}\mathbf{V}_{\mathbf{0}})$$

$$\mathbf{V}_i^{\mathbf{o}} = s_i\boldsymbol{\mu}_i^{\mathbf{o}} + \hat{\mathbf{s}}_i\rho_i^{\mathbf{o}}$$

Sampling Strategy

- Remembering that $\mathbf{V}_i$ is just a stand-in for $\boldsymbol{\mu}_i$ and $V_i^{\parallel}$, the variables we are actually using to express the posterior distribution, we mechanically substitute (e.g., integrate over the delta-function) to give

$$\hat{\mathbf{s}}_i{}' \mathbf{V}_i \rightarrow V_i^{\parallel},$$

$$(\mathbf{V}_i - \hat{\mathbf{s}}_i V_i^{\parallel}) / s_i \rightarrow \boldsymbol{\mu}_i$$

Sampling Strategy

- We sample the U_i using a Metropolis-Hastings step. Our proposal is $U_i^* \sim N(0, w)$ with an appropriate w, adjusted for good mixing. The conditional is proportional to

$$s_i^2 N(s_i \boldsymbol{\mu}_i + \hat{\mathbf{s}}_i V_i^{\parallel} - \mathbf{V_0}, \mathbf{W})$$

with the informative prior on U_i we described earlier,

$$U_i \sim N(0, (0.15)^2)$$

- In calculating the conditional, as always, we mechanically substitute the relationship between s_i and M_i.

Sampling Strategy

- We sample on M using a Metropolis-Hastings step. Our proposal for M^* is a t distribution centered on M with an appropriate choice of degrees of freedom and variance, adjusted for good mixing. The conditional is proportional to

$$\prod s_i^2 N(s_i \boldsymbol{\mu}_i + \hat{\mathbf{s}}_i V_i^{\parallel} - \mathbf{V_0}, \mathbf{W})$$

  with a prior on M that may or may not be informative (as discussed earlier).
- Again, we mechanically substitute where appropriate.
- We found that *dof*=10 and variance=0.01 did well.

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Results on *M*

- Our key astrophysically interesting result is *M*.
  - Plot of samples on *M*
  - Histogram of *M*
  - Mean, variance of *M*

Results on *M*

- Hawley et. al. used a maximum likelihood technique to study this problem. For comparison, we used their data.
- They ran cases with a number of subsets of stars, broken down by various criteria. We look only at the "ab" stars (normal pulsators, N=141) and the "c" stars (overtone pulsators, N=17).

	M (ab stars)
This study	0.79 ± 0.11
Hawley *et. al.*	0.76 ± 0.14
	M (c stars)
This study	0.75 ± 0.27
Hawley *et. al.*	1.09 ± 0.38

Other Methods for *M*

- The data of Hawley *et. al.* had some incorrect apparent magnitudes. Reanalysis with corrected data (not available to us for this study) shows that the value of *M* should be decreased by approximately 0.08 magnitudes.
- The best direct measurement of the distance of an RR Lyrae star, by Hubble Space Telescope, gives $M=0.61\pm0.10$ magnitudes.
- The results of Skillen *et. al.*, using the Surface Brightness method (similar to the method used by Barnes *et. al.* and reported at ISBA 2000) give $M=0.65\pm0.10$ magnitudes.

Results

- Other results of interest are
  - The solar motion $\mathbf{V}_0$ relative to the sample. This informs us of the relative motion of the Sun through the ensemble of RR Lyrae stars in the sample (Units are km/sec).

ab stars	V_ϖ	V_θ	V_z
This study	-12 ± 11	-136 ± 9	-8 ± 7
Hawley *et. al.*	-10 ± 13	-155 ± 12	-9 ± 8
c stars			
This study	-26 ± 25	-113 ± 26	-2 ± 12
Hawley *et. al.*	-26 ± 25	-124 ± 25	-6 ± 13

Results

- Other results of interest are
  - Velocity Ellipsoid: This tells us how the galactocentric orbits of the RR Lyrae stars are oriented in a statistical sense, as described by the covariance matrix of the velocities. In galactic cylindrical coordinates (ϖ,θ,z) the matrix is believed roughly diagonal, with the on-diagonal dispersions decreasing from ϖ to θ to z. (Units are km/sec, see next chart).
  - There is a significant difference in the estimated standard deviations between this study and Hawley *et. al*. Hawley has stated (private communication) that she is not confident of her estimates.

Results

- Other results of interest are
  - Velocity Ellipsoid

ab stars	σ_{ϖ}	σ_{θ}	σ_{z}
This Study	133 ± 9	109 ± 7	75 ± 5
Hawley *et. al.*	150 ± 59	120 ± 47	87 ± 33
c stars			
This Study	101 ± 22	106 ± 23	46 ± 11
Hawley *et. al.*	101 ± 57	71	51

Results

- Compare with
  - Hawley, *et. al.*
    - The sample of "ab" stars agrees within the errors with the analysis of Hawley *et. al.* However, the "c" sample gives a discrepant value of *M*. Their value is high, compared to ours.
    - The reason for this is unknown, but in their analysis of these data they did not solve for the full covariance matrix of the velocities. Instead, they set the off-diagonal terms to zero, and fixed the ratios for the on-diagonal terms to that given by the reduction of the larger "ab" data.

Sampling on U_i

- Dominated by prior, it seems that the posterior on each U_i simply follows the prior
  - It seems that the U_i are unidentified in this model
- J. Berger raised concerns about leaving it out, so we put it in; results essentially unchanged

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Future Research

- We want to investigate the effect of metallicity on the results. It is known that *M* depends on this, but we want to see if our data can detect this effect
- There is a much larger and better sample of proper motions from the HIPPARCOS satellite, which together with new radial velocities should provide better results.